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THE WEAK AND STRONG GAUSSIAN PROBABILISTIC REALIZATION PROBLEM

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The weak and strong Gaussian probabilistic realization problem *)

by

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ABSTRACT

A classification is given of all σ-algebras that make two given σ-algebras conditionally independent in the case that the σ-algebras are generated by finite dimensional Gaussian random variables. In addition a classification is given of all Gaussian measures that have the conditional independence property and such that, restricted to a subspace, they coincide with a given measure.

KEY WORDS & PHRASES: *Conditional independence, Gaussian random variables, canonical variable representation, sufficient statistics, stochastic systems*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

The purpose of this paper is to present the solution of the weak and strong probabilistic realization problem for σ-algebras generated by finite dimensional Gaussian random variables.

The stochastic realization problem in stochastic system theory is to construct stochastic dynamic system representations for stochastic processes. There is a growing literature on this subject [4,5,7,9] mainly for Gaussian processes. The problem is still not satisfactorily solved. One open question in the Gaussian case is the explicit classification of all minimal stochastic realizations. In a static setting the stochastic realization problem reduces to the probabilistic realization problem, to be formulated below. In this paper this problem will be solved. The solution given may provide insight in the classification of minimal Gaussian stochastic realizations.

The main concept in stochastic realization theory, as shown in [7,10], is the conditional independence relation for σ-algebras. This relation is a key-property in many areas of probability theory and stochastic processes. Examples of such areas are sufficient statistics, Markov processes, information theory, random fields and stochastic system theory.

What is the problem here? Assume given two jointly Gaussian random variables and consider the σ-algebras that they generate. One may ask for all the σ-algebras that make these two σ-algebras conditional independent. To exclude some trivial answers the concept of a minimal σ-algebra that makes these two σ-algebras conditional independent must be introduced. The strong probabilistic realization problem is then to show existence of σ-algebras that make two given σ-algebras minimal conditional independent, to classify all such σ-algebras and to develop an algorithm that constructs these σ-algebras. The weak probabilistic realization problem is analogous to the above problem, except that in this case the underlying probability space may be constructed. A still open problem is the probabilistic realization problem in the case that the σ-algebras are arbitrary, not necessarily generated by Gaussian random variables.

The approach of the paper is a mixture of probabilistic and geometric analysis. The main objects of the paper are σ-algebras generated by finite dimensional Gaussian random variables. From NEVEU [6] it is clear that a Hilbert-space framework may be used in this case. This approach has been followed in [5]. However, this line of work is insufficient for the problem to be considered here. Because of the restriction to σ-algebras generated by finite dimensional Gaussian random variables a more explicit classification may be obtained.

A brief summary of the paper follows. The problem formulation is given in the next section, while some preliminaries are presented in Section 3. The weak probabilistic realization problem is solved in Section 4 and the strong probabilistic realization problem in Section 5.

2. PROBLEM FORMULATION

In this section some notation is introduced and the problem defined.

In this paper (Ω,F,P) denotes a complete probability space, consisting of a set Ω, a σ-algebra F, and a probability measure P. Let

$$\underline{F} = \{G \text{ a } \sigma\text{-algebra of elements of } \Omega \mid G \subset F, \text{ completed with all the null-sets of } F\},$$

and for $G \in \underline{F}$

$$L^+(G) = \{x\colon \Omega \to R_+ \mid x \text{ is } G\text{-measurable}\}.$$

If $y\colon \Omega \to R^k$ is a random variable, then $F^y = \sigma(\{y\}) \in \underline{F}$ is the σ-algebra generated by y. If $F_1,F_2 \in \underline{F}$, then $(F_1 \vee F_2) \in \underline{F}$ denotes the smallest σ-algebra that contains both F_1 and F_2. The notation $(F_1,F_2) \in I$ is used to indicate that F_1,F_2 are independent σ-algebras. R^n will be equiped with the σ-algebra of the Lebesgue measurable sets, together denoted by (R^n,B_n).

2.1. <u>DEFINITION</u>. The *conditional independence relation* for a triple of σ-algebras $F_1,F_2,G \in \underline{F}$ is defined by the condition that for all $y_1 \in L^+(F_1)$, $y_2 \in L^+(F_2)$

$$E[y_1y_2|G] = E[y_1|G]E[y_2|G].$$

Equivalently, if

$$E[y_1 | F_2 \vee G] = E[y_1 | G]$$

for all $y_1 \in L^+(F_1)$. Then one says that F_1, F_2 are *conditional independent* given G. Notation $(F_1, G, F_2) \in CI$. □

The equivalence follows from [1, II.45].

Some notation will be introduced. Let

$$Z_+ = \{1,2,3,\dots\}, \quad N = \{0,1,2,\dots\},$$

and for $n \in Z_+$

$$Z_n = \{1,2,\dots,n\}, \quad N_n = \{0,1,2,\dots,n\}.$$

If $n \in Z_+$, $Q \in R^{n\times n}$, then Q^T denotes the transposed of Q; if Q is symmetric, then $Q \geq 0$ denotes that Q is positive definite and $Q > 0$ that it is strictly positive definite.

A finite dimensional Gaussian random variable with parameters $n \in Z_+$, $\mu \in R^n$, $Q \in R^{n\times n}$, satisfying $Q = Q^T \geq 0$, is a random variable $x\colon \Omega \to R^n$ such that for all $u \in R^n$

$$E[\exp(iu^T x)] = \exp(iu^T\mu - \tfrac{1}{2}u^T Q u).$$

Notation: $x \in G(\mu,Q)$; $(x_1,\dots,x_m) \in G(\mu,Q)$ will denote that with $x^T = (x_1^T,\dots,x_m^T)$, $x \in G(\mu,Q)$. If $x \in G$, then Q_{xx} may denote its covariance matrix. The notation $G(\mu,Q)$ will stand for a Gaussian measure on (R^n, B_n) as indicated above, when n is clear from the context.

2.2. <u>DEFINITION</u>. The *Gaussian conditional independence relation* for a triple of σ-algebras $F^{y1}, F^x, F^{y2} \in \underline{F}$ generated by $y_1\colon \Omega \to R^{k1}$, $y_2\colon \Omega \to R^{k2}$, $x\colon \Omega \to R^n$, is defined by the conditions

1. $(F^{y1}, F^x, F^{y2}) \in CI$;
2. $(y_1, x, y_2) \in G$.

Notation: $(F^{y1}, F^x, F^{y2}) \in CIG$. □

Given $(y_1,y_2) \in G$ there exists a random variable x such that $(F^{y_1},F^x,F^{y_2}) \in CIG$. For example, $x = y_1$, or $x = y_2$, are such random variables. From many viewpoints it is of interest to ask for a minimal dimension for the random variable x.

2.3. DEFINITION. The *minimal Gaussian conditional independence relation* for a triple of σ-algebras $F^{y_1},F^{y_2},F^x \in \underline{F}$ generated by $y_1: \Omega \to R^{k_1}$, $y_2: \Omega \to R^{k_2}$, $x: \Omega \to R^n$ is defined by the conditions

1. $(F^{y_1},F^x,F^{y_2}) \in CIG$;
2. if $F^{x_1} \in \underline{F}$, $F^{x_1} \subset F^x$, $(F^{y_1},F^{x_1},F^{y_2}) \in CIG$, and $(y_1,y_2,x,x_1) \in G$, then $F^{x_1} = F^x$.

Then one says that F^x makes F^{y_1},F^{y_2} minimal Gaussian conditional independent. Notation: $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$. □

2.4. PROBLEM. The *weak Gaussian probabilistic realization problem* for a Gaussian measure on $(R^{k_1+k_2},B_{k_1+k_2})$ is:

a. to show existence of a state space (R^n,B_n) and of a Gaussian measure P on $(R^{k_1+k_2+n},B_{k_1+k_2+n})$ such that, after introduction of the canonical variables,
   1. $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$;
   2. the measure on (y_1,y_2) coincides with the given measure;

   the triple (R^n,B_n,P) will then be called a *minimal weak Gaussian probabilistic realization* of the given measure;
b. to classify all such minimal realizations;
c. to develop an algorithm that constructs all such minimal realizations. □

2.5. PROBLEM. The *strong Gaussian probabilistic realization problem* for a triple of Gaussian random variables (y_1,y_2,v) is:

a. to show existence of triples (R^n,B_n,F^x), where $x: \Omega \to R^n$, such that
   1. $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$;
   2. $F^x \subset F^{y_1} \vee F^{y_2} \vee F^v$ and $(y_1,y_2,v,x) \in G$;

   such a triple will then be called a *minimal strong Gaussian probabilistic realization* of the given triple;
b. to classify all such minimal realizations;
c. to develop an algorithm that constructs all such minimal probabilistic realizations. □

A technical result on conditional independence is needed that will be proved next. Note that if $(F_1,G,F_2) \in CI$ and $F_3 \subset F_1$, then $(F_3,G,F_2) \in CI$.

2.6. PROPOSITION. *If* $F_1,F_2,G_1,G_2 \in \underline{F}$, $(F_1,G_1 \vee G_2,F_2) \in CI$, *and* $(G_2,F_1 \vee F_2 \vee G_1) \in$ $\in I$, *then* $(F_1,G_1,F_2) \in CI$.

PROOF. With $g_2 \in L^+(G_2)$ and $(G_2,F_1 \vee F_2 \vee G_1) \in I$

$$E[g_2|F_1 \vee G_1] = E[g_2] = E[g_2|G_1],$$

hence $(F_1,G_1,G_2) \in CI$. Let $y_1 \in L^+(F_1)$. Then

$$\begin{aligned} E[y_1|F_2 \vee G_1] &= E[E[y_1|F_2 \vee G_1 \vee G_2]|F_2 \vee G_1] \\ &= E[E[y_1|G_1 \vee G_2]|F_2 \vee G_1] \\ &= E[E[y_1|G_1]|F_2 \vee G_1] = E[y_1|G_1]. \end{aligned}$$

One concludes with the equivalent condition of 2.1. □

Finally some additional notation for matrices is introduced. For $n \in Z_+$ let

$$\begin{aligned} D_n &= \{A \in R^{n\times n} \mid A \text{ diagonal}\}, \\ D_n^+ &= \{A \in D_n \mid A \geq 0\}, \\ D_n^{0+} &= \{A \in D_n^+ \mid \text{if } A = \text{diag}(a_1,\ldots,a_n) \text{ then } a_1 \geq a_2 \geq \ldots \geq a_n\}, \\ O_n &= \{S \in R^{n\times n} \mid SS^T = I = S^TS\}, \end{aligned}$$

the set of orthogonal matrices. For $A \in R^{n\times n}$ let

$$C_n(A) = \{(S_1,S_2) \in O_n \times O_n \mid S_2^TS_1A = AS_2^TS_1\}.$$

It is easily verified that $C_n(A)$ is an equivalence relation. The set of equivalence classes of O_n over $C_n(A)$ is denoted by $O_n/C_n(A)$. The class of matrices that commute with a given matrix is described in [2, 1.VIII 2].

3. PRELIMINARIES

In this section the canonical variable form for Gaussian random variables is introduced and an equivalent condition for CIG_{min} is derived. These preliminaries will be used in the following sections.

To describe the relationship between two random variables HOTELLING [3] has introduced the concept of a canonical variable form. For Gaussian random variables this form has a rather explicit structure that is stated below.

3.1. DEFINITION. Given $y_1\colon \Omega \to R^{k}$, $y_2\colon \Omega \to R^{k_2}$, $(y_1,y_2) \in G(0,K)$. These random variables are said to be in *canonical variable form* if

$$K = \left(\begin{array}{ccc|ccc} I & & & I & & \\ & I & & & \Lambda & \\ & & I & & & 0 \\ \hline I & & & I & & \\ & \Lambda & & & I & \\ & & 0 & & & I \end{array}\right) \in R^{(k_1+k_2)\times(k_1+k_2)},$$

where $\Lambda \in D^{0+}_{k_{12}}$, $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_{k_{12}})$, $1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k_{12}} > 0$. Compatible with this decomposition let $y_{11}\colon \Omega \to R^{k_{11}}$, $y_{12}\colon \Omega \to R^{k_{12}}$, $y_{13}\colon \Omega \to R^{k_{13}}$, $y_{21}\colon \Omega \to R^{k_{11}}$, $y_{22}\colon \Omega \to R^{k_{12}}$, $y_{23}\colon \Omega \to R^{k_{23}}$, $y_1^T = (y_{11}^T,y_{12}^T,y_{13}^T)$, $y_2^T = (y_{21}^T,y_{22}^T,y_{23}^T)$. Furthermore, let

$$\Sigma = \begin{pmatrix} I & \Lambda \\ \Lambda & 0 \end{pmatrix} \in R^{2k_{12}\times 2k_{12}}, \quad W = \begin{pmatrix} I & & \\ & \Lambda & \\ & & 0 \end{pmatrix} \in R^{k_1\times k_2}. \qquad \square$$

It is a classical result [8] that for any pair $(z_1,z_2) \in G(0,K_1)$, with $K_1 > 0$, there exists a basis transformation $(z_1,z_2) \mapsto (S_1z_1,S_2z_2)$ such that (S_1z_1,S_2z_2) is in canonical variable form. On the basis of the canonical variable form one may formulate a canonical form for Gaussian measures.

The problem posed in 2.4 is the construction and classification of σ-algebras that make two given σ-algebras minimal Gaussian conditional independent. This problem is analogous to the construction of realizations in linear system theory. There it is known that a dynamical system has a state space of minimal dimension iff the dynamical system is observable and controllable. Furthermore, all minimal realizations are equivalent in a well defined sense. What remains of this picture in probabilistic realization? The concept of probabilistic observability will be defined first.

3.2. <u>DEFINITION</u>. Given $(F^{y_1}, F^x, F^{y_2}) \in CIG$. This triple will be called *probabilistic observable* if the map $x \mapsto E[\exp(iu^T y_1) | F^x]$ is injective on the support of x. It will be called *probabilistic reconstructible* if the map $x \mapsto E[\exp(iu^T y_2) | F^x]$ is injective on the support of x. □

Suppose that through multiple experiments one is able to obtain an estimate of the measure of y_1 for a fixed value of x. Then probabilistic observability implies that from this measure one can determine the value of the state x uniquely. This property motivates the above definition of probabilistic observability.

With $(y_1, x, y_2) \in G(0,Q)$ and a basis for x such that $Q_{xx} > 0$ one has that

$$E[\exp(iu^T y_1) | F^x] = \exp(iu^T Q_{y_1x} Q_{xx}^{-1} x - \tfrac{1}{2} u^T [Q_{y_1y_1} - Q_{y_1x} Q_{xx}^{-1} Q_{xy_1}] u).$$

Thus $(F^{y_1}, F^x, F^{y_2}) \in CIG$ is probabilistic observable iff $\mathrm{rank}(Q_{y_1x}) = \mathrm{rank}(Q_{xx})$. The following result, an equivalence condition for CIG_{min}, is now motivated.

3.3. <u>THEOREM</u>. *Given* $y_1: \Omega \to R^{k_1}$, $y_2: \Omega \to R^{k_2}$, $x: \Omega \to R^n$. *The following are equivalent:*

(a) $(F^{y_1}, F^x, F^{y_2}) \in CIG_{min}$;

(b) 1. $(y_1, x, y_2) \in G$;

2. $Q_{y_1y_2} = Q_{y_1x} Q_{xx}^{-1} Q_{xy_2}$;

3. $\mathrm{rank}(Q_{y_1x}) = \mathrm{rank}(Q_{xx}) = \mathrm{rank}(Q_{y_2x})$.

Here it has been assumed that a basis has been chosen such that $Q_{xx} > 0$.

The proof of 3.3 is based on the following intermediate results.

3.4. PROPOSITION. *Given* $y_1\colon \Omega \to R^{k_1}$, $y_2\colon \Omega \to R^{k_2}$, $x\colon \Omega \to R^n$, $(y_1,x,y_2) \in G$. *Suppose that a basis for* x *has been chosen such that* $Q_{xx} > 0$. *Then the following are equivalent:*

(a) $(F^{y_1},F^x,F^{y_2}) \in CIG$;

(b) $Q_{y_1y_2} = Q_{y_1x}Q_{xx}^{-1}Q_{xy_2}$.

PROOF. This is a calculation via the conditional characteristic function. □

3.5. PROPOSITION. *Let* y_1,y_2,x *be as defined in* 3.4. *If* $(F^{y_1},F^x,F^{y_2}) \in CIG$,

$$x_1\colon \Omega \to R^{k_1}, \qquad x_1 = E[y_1|F^x],$$
$$x_2\colon \Omega \to R^{k_2}, \qquad x_2 = E[y_2|F^{x_1}],$$

then $(F^{y_1},F^{x_2},F^{y_2}) \in CIG$ *and* $\mathrm{rank}(Q_{x_2x_2}) = \mathrm{rank}(Q_{y_1y_2})$.

PROOF. Because $F^{x_1} \subset F^x$,

$$\begin{aligned}E[y_1|F^{x_1}vF^{y_2}] &= E[E[y_1|F^xvF^{y_2}]|F^{x_1}vF^{y_2}]\\ &= E[E[y_1|F^x]|F^{x_1}vF^{y_2}] = E[y_1|F^x]\\ &= E[E[y_1|F^x]|F^{x_1}] = E[y_1|F^{x_1}]\end{aligned}$$

and with 2.1, $(F^{y_1},F^{x_1},F^{y_2}) \in CIG$. Similarly, using $F^{x_2} \subset F^{x_1}$, one obtains that $(F^{y_1},F^{x_2},F^{y_2}) \in CIG$. Then $x_1 = E[y_1|F^x] = Q_{y_1x}Q_{xx}^{-1}x$. Let $n_1 = \mathrm{rank}(Q_{x_1x_1})$. Then there exists $S \in R^{n_1\times k_1}$ such that

$$I_{n_1} = SQ_{x_1x_1}S^T = SQ_{y_1x}Q_{xx}^{-1}Q_{xy_1}S^T.$$

Then $F^{x_1} = F^{Sx_1}$ and

$$x_2 = E[y_2F^{Sx_1}] = Q_{y_2x_1}S^TQ_{Sx_1,Sx_1}^{-1}Sx_1 = Q_{y_2x_1}S^TSQ_{y_1x}Q_{xx}^{-1}x,$$

$$Q_{x_2x_2} = Q_{y_2y_1}S^TSQ_{y_1y_2}.$$

Let $n_2 = \text{rank}(Q_{x_2x_2})$. Then there exists $R \in R^{n_2 \times k_2}$, such that

$$I_{n_2} = RQ_{x_2x_2}R^T = RQ_{y_2y_1}S^TSQ_{y_1y_2}R^T.$$

Define $w: \Omega \to R^{n_2}$, $w = Rx_2$. Then $Q_{ww} = I_{n_2}$, $F^{x_2} = F^w$, and $\text{rank}(Q_{ww}) \leq \text{rank}(Q_{y_1y_2})$. By $(F^{y_1},F^w,F^{y_2}) = (F^{y_1},F^{x_2},F^{y_2}) \in$ CIG and 3.4 $\text{rank}(Q_{ww}) \geq \text{rank}(Q_{y_1y_2})$. The conclusion then follows. □

PROOF of 3.3. If $Q_{y_1y_2} = Q_{y_1x}Q_{xx}^{-1}Q_{xy_2}$, then $\text{rank}(Q_{y_1x}) = \text{rank}(Q_{xx}) = \text{rank}(Q_{y_2x})$ iff $\text{rank}(Q_{y_1y_2}) = \text{rank}(Q_{xx})$. a → b. By 3.4(b) $\text{rank}(Q_{y_1y_2}) \leq \text{rank}(Q_{xx})$. Suppose that $\text{rank}(Q_{y_1y_2}) < \text{rank}(Q_{xx})$. As in 3.5 define $x_1 = E[y_1|F^x]$, $x_2 = E[y_2|F^{x_1}]$, and conclude that $(F^{y_1},F^{x_2},F^{y_2}) \in$ CIG and $\text{rank}(Q_{x_2x_2}) = \text{rank}(Q_{y_1y_2}) < \text{rank}(Q_{xx})$. This and $F^{x_2} \subset F^x$ contradicts the minimality. Thus $\text{rank}(Q_{y_1y_2}) = \text{rank}(Q_{xx})$ and one concludes with the above remark. b → a. Let $F^{x_1} \subset F^x$, $(F^{y_1},F^{x_1},F^{y_2}) \in$ CIG, and $(y_1,y_2,x,x_1) \in G$. Suppose that a basis for x_1 has been chosen such that $Q_{x_1x_1} > 0$, $n_1 = \text{rank}(Q_{x_1x_1})$. By $(y_1,y_2,x,x_1) \in G$ and $F^{x_1} \subset F^x$ one has $n_1 \leq n$. Furthermore $(F^{y_1},F^{x_1},F^{y_2}) \in$ CIG implies by 3.4 that $n = \text{rank}(Q_{xx}) = \text{rank}(Q_{y_1y_2}) \leq \text{rank}(Q_{x_1x_1}) = n_1$, hence $n_1 = n$. Familiar arguments now yield that $F^x = F^{x_1}$ and $(F^{y_1},F^x,F^{y_2}) \in \text{CIG}_{\min}$. □

Note that the construction procedure given in 3.5 provides a way to construct σ-algebras that have the minimality property. Based on the decomposition presented in 3.1 some results may be obtained that will facilitate the proofs in the following two sections. This is done below.

3.6. PROPOSITION. *Given* $y_1: \Omega \to R^{k_1}$, $y_2: \Omega \to R^{k_2}$, $(y_1,y_2) \in G(0,K)$ *with* K *in the canonical variable form as given in* 3.1. *The notation of* 3.1 *is adopted. Then* $(F^{y_1},F^x,F^{y_2}) \in \text{CIG}_{\min}$ *iff there exists a basis for* x *such that* $x^T = (x_1^T,x_2^T)$, $x_1: \Omega \to R^{k_{11}}$, $x_2: \Omega \to R^{k_{12}}$, $(F^{y_{11}},F^{x_1},F^{y_{21}}) \in \text{CIG}_{\min}$ *and* $(F^{y_{12}},F^{x_2},F^{y_{22}}) \in \text{CIG}_{\min}$.

PROOF. Necessity of the decomposition. By the remark preceding 2.6 and $(F^{y_1},F^x,F^{y_2}) \in \text{CIG}_{\min}$ it follows that

$$(F^{y_{11}} \vee F^{y_{12}},F^x,F^{y_{21}} \vee F^{y_{22}}) \in \text{CIG}.$$

Define $x_1: \Omega \to R^n$, $x_2: \Omega \to R^n$,

$$x_1 = E[x|F^{y_{11}}], \quad x_2 = x - x_1.$$

Then $(F^{x_2}, F^{y_{11}}) \in I$. Because of $(F^{y_{11}}, F^x, F^{y_{21}}) \in CIG$, $y_{11} = y_{21}$ and 2.1, one obtains

$$E[y_{11}y_{11}^T|F^x] = E[y_{11}|F^x]E[y_{11}|F^x],$$

hence $y_{11} = E[y_{11}|F^x]$. Thus $F^{x_1} \subset F^{y_{11}} \subset F^x$. Furthermore, $x_2 = x - x_1$ implies that $F^{x_2} \subset (F^x \vee F^{x_1}) \subset F^x$, hence $F^{x_1} \vee F^{x_2} = F^x$. Now $(F^{y_{11}} \vee F^{y_{12}}, F^{x_1} \vee F^{x_2}, F^{y_{21}} \vee F^{y_{22}}) \in CIG$, $(F^{x_2}, F^{y_{11}}) \in I$, $F^{x_1} \subset F^{y_{11}}$, and 2.6 imply that $(F^{y_{11}}, F^{x_1}, F^{y_{21}}) \in CIG$ and $(F^{y_{12}}, F^{x_2}, F^{y_{22}}) \in CIG$. Then

$$\begin{aligned} k_{11} + k_{12} &= \mathrm{rank}(Q_{xx}) = \mathrm{rank}(Q_{x_1x_1}) + \mathrm{rank}(Q_{x_2x_2}) \\ &\geq \mathrm{rank}(Q_{y_{11}y_{21}}) + \mathrm{rank}(Q_{y_{12}y_{22}}) \\ &= k_{11} + k_{12}, \end{aligned}$$

where the inequality follows from 3.4(b). Hence equality holds throughout, $\mathrm{rank}(Q_{x_1x_1}) = k_{11}$, $\mathrm{rank}(Q_{x_2x_2}) = k_{12}$ and the conclusion follows. Finally one may reduce the effective dimensions of x_1, x_2. The sufficiency of the decomposition is a verification using 3.3. □

3.7. PROPOSITION. *Given* $y: \Omega \to R^k$, $y \in G(0,I)$, $x: \Omega \to R^n$. *Then* $(F^y, F^x, F^y) \in$ $\in CIG_{min}$ *iff with respect to some basis* $x = y$ *a.s.*

PROOF. The elementary proof is omitted. □

4. THE WEAK GAUSSIAN PROBABILISTIC REALIZATION PROBLEM

In this section the weak Gaussian probabilistic realization problem is solved.

4.1. DEFINITION. Let a Gaussian measure $G(0,K)$ be given on $(R^{k_1+k_2}, B_{k_1+k_2})$. Define the set of *weak Gaussian probabilistic realizations*

$$\mathrm{WPR}(R^{k_1+k_2}, B_{k_1+k_2}), G(0,K))$$

$$= \{(R^n, B_n),\ P\colon B_{k_1+k_2+n} \to [0,1] \mid \text{if } \Omega = R^{k_1+k_2+n},$$

$$F = B_{k_1+k_2+n},\ P \text{ a probability measure},$$

$$y_1\colon \Omega \to R^{k_1},\ y_1((\omega_1,\omega_2,\omega_3)) = \omega_1,$$

$$y_2\colon \Omega \to R^{k_2},\ y_2((\omega_1,\omega_2,\omega_3)) = \omega_2,$$

$$x\colon \Omega \to R^n,\ x((\omega_1,\omega_2,\omega_3)) = \omega_3,$$

$$\text{then } (F^{y_1}, F^x, F^{y_3}) \in \mathrm{CIG}_{min}, \text{ and } (y_1,y_2) \in G(0,K)\}.$$

The elements (R^{n_1}, B_{n_1}, P_1), $(R^{n_2}, B_{n_2}, P_2) \in \mathrm{WPR}$ are said to be equivalent if $P_1 = P_2$ up to a basis transformation of the underlying probability space. In the following the set WPR will be identified with the set of equivalence classes obtained by dividing out the above equivalence relation.

In the above definition y_1, y_2, x are called canonical variables, which term must be distinguished from the canonical variable form defined in 3.1.

4.2. THEOREM. *Given the set* $\mathrm{WPR}(R^{k_1+k_2}, B_{k_1+k_2}, G(0,K))$ *where a basis has been chosen such that the matrix* K *has the form as given in* 3.1. *The notation of* 3.1 *is adopted. Let*

$$\mathrm{WPA} = \{U \in D^{0+}_{k_{12}},\ S \in O_{k_{12}} \mid S \in O_{k_{12}}/C_{k_{12}}(U),$$

$$U = \mathrm{diag}(u_1,\ldots,u_{k_{12}}),\ 1 \geq u_1 \geq u_2 \geq \ldots \geq u_{k_{12}} \geq 0\}.$$

Define the map r: WPA → WPR *by*

$$r(U,S) = (R^n, B_n, G(0,L)),$$

where

$$n = k_{11} + k_{12}, \quad (\Lambda^{-1} - \Lambda)^{\frac{1}{2}} \in D^+_{k_{12}},$$

$$Q = (\Lambda^{-1} - \Lambda)^{\frac{1}{2}} \, SUS^T (\Lambda^{-1} - \Lambda)^{\frac{1}{2}} + \Lambda \in R^{k_{12} \times k_{12}},$$

$$L = \left(\begin{array}{ccc|ccc|cc}
I & & & I & & & I & \\
& I & & & \Lambda & & 0 & \Lambda^{\frac{1}{2}} \\
& & I & & & 0 & 0 & 0 \\
\hline
I & & & I & & & I & 0 \\
& \Lambda & & & I & & 0 & \Lambda^{\frac{1}{2}}Q \\
& & 0 & & & I & 0 & 0 \\
\hline
I & 0 & 0 & I & 0 & 0 & I & \\
0 & \Lambda^{\frac{1}{2}} & 0 & 0 & Q\Lambda^{\frac{1}{2}} & & & Q
\end{array} \right) \in R^{(k_1+k_2+n)\times(k_1+k_2+n)}.$$

Then r: WPA → WPR *is well defined and a bijection.*

The solution to the weak Gaussian probabilistic realization problem is given by 4.2, since it classifies all minimal weak Gaussian probabilistic realizations and provides an algorithm to construct these realizations. The structure of the solution will be explained in Section 5.

4.3. LEMMA. *Given the set* $\mathrm{WPR}(R^{2k}, B_{2k}, G(0,\Sigma))$ *with* Σ *as given in* 3.1. *Let*

$$\underline{Q} = \{Q \in R^{k\times k} \mid Q = Q^T \geq 0,\ A = (\Lambda^{-1} - \Lambda)^{-1},\ Q + Q\Lambda A + A\Lambda Q - QAQ - A \geq 0 \quad \}.$$

Define $r_1 \colon \underline{Q} \to \mathrm{WPR}$ *by* $r_1(Q) = (R^k, B_k, G(0,L))$, *where*

$$L_1 = \begin{pmatrix} I & \Lambda & \Lambda^{\frac{1}{2}} \\ \Lambda & I & \Lambda^{\frac{1}{2}}Q \\ \Lambda^{\frac{1}{2}} & Q\Lambda^{\frac{1}{2}} & Q \end{pmatrix} \in R^{3k\times 3k}.$$

Then the map $r_1 \colon \underline{Q} \to \mathrm{WPR}(R^{2k}, B_{2k}, G(0,\Sigma))$ *is well defined and a bijection.*

Some calculations needed in the proof of 4.3 are summarized in the following result.

4.4. PROPOSITION. *Given* $\Lambda \in R^{n\times n}$, *as defined in* 3.1, $Q \in R^{n\times n}$, *and* $L_1 \in R^{3n\times 3n}$ *as defined in* 4.3. *Assume that* $Q = Q^T$. *Then the following are equivalent:*

a. $L_1 \geq 0$;
b. $Q \in \underline{Q}$, *where* $\underline{Q}$ *is as defined in* 4.3;
c. $\Lambda \leq Q \leq \Lambda^{-1}$.

PROOF. Σ, as defined in 3.1, is nonsingular and

$$\Sigma^{-1} = \begin{pmatrix} (I-\Lambda^2)^{-1} & -A \\ -A & (I-\Lambda^2)^{-1} \end{pmatrix}.$$

Elementary row and column operations now yield that $L_1 \geq 0$ iff

$$Q \geq \Lambda \quad \text{and} \quad I - \Lambda^{\frac{1}{2}} Q \Lambda^{\frac{1}{2}} \geq 0,$$

iff

$$Q - (\Lambda^{\frac{1}{2}} \quad Q\Lambda^{\frac{1}{2}})\Sigma^{-1}\begin{pmatrix} \Lambda^{\frac{1}{2}} \\ \Lambda^{\frac{1}{2}}Q \end{pmatrix} \geq 0.$$

A calculation then gives the result. □

PROOF of 4.3.

1. It will be shown that r_1 is well defined. If $Q \in \underline{Q}$ then it follows from 4.4 that $L_1 = L_1^T \geq 0$ and $Q > 0$. Then $G(0,L_1)$ is a valid Gaussian measure on (R^{3k},B_{3k}). Denote the canonical variables by (y_1,y_2,x). Furthermore,

$$Q_{y_1x}Q_{xx}^{-1}Q_{xy_2} = \Lambda^{\frac{1}{2}}Q^{-1}Q\Lambda^{\frac{1}{2}} = \Lambda = Q_{y_1y_2},$$

$$\mathrm{rank}(Q_{y_1x}) = \mathrm{rank}(\Lambda) = \mathrm{rank}(Q_{xx}) = \mathrm{rank}(Q\Lambda^{\frac{1}{2}}) = \mathrm{rank}(Q_{y_2x}).$$

By 3.3 one obtains that $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$, and by definition of L_1 $(y_1,y_2) \in G(0,\Sigma)$.

2. Surjectiveness. Because of the minimality in $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$ and 3.3 one may choose a basis for x of dimension k; then $Q_{xx} > 0$. From $(F^{y_1},F^x,F^{y_2}) \in CIG$ it follows that $\Lambda = Q_{y_1x}Q_{xx}^{-1}Q_{xy_2}$, hence Q_{y_1x} is non-singular. Let $x_1 = \Lambda^{\frac{1}{2}}Q_{y_1x}^{-1}x$ and $Q = Q_{x_1x_1}$. Then $Q_{y_1x_1} = \Lambda^{\frac{1}{2}}$, $Q_{y_2x_1} = \Lambda^{\frac{1}{2}}Q$ and $(y_1,y_2,x_1) \in G(0,L_1)$ with L_1 as given in 4.3. Because $L_1 = L_1^T \geq 0$, it follows from 4.4 that $Q \in \underline{Q}$.

3. Injectiveness. As indicated in 2. above, one may choose a basis for the probability space, such that with respect to this basis the corresponding covariance matrix has the form L_1. Hence $(R^{n_1},B_{n_1},P_1) = (R^{n_2},B_{n_2},P_2) \in WPR$ implies that $Q_1 = Q_2 \in \underline{Q}$. □

PROOF of 4.2.

1. It will be shown that the map r is well defined. Given $(U,S) \in WPA$ it is a calculation to verify that $Q \in \underline{Q}$. Using 4.4 and linear algebra operations one obtains that $L = L^T \geq 0$. The Gaussian measure $G(0,L)$ is thus well defined. Denote the canonical variables, as defined in 4.1, by (y_1,y_2,x). Then $(y_1,y_2) \in G(0,K)$. Using the expression for L and 3.3, one concludes that $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$, and $(R^n,B_n,G(0,L)) \in WPR$.

2. Surjectiveness. Because of the minimality in $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$ and 3.3, one may take a basis for x of dimension $k_{11}+k_{12}$. By 3.6 there exists $x_1: \Omega \to R^{k_{11}}$, $x_2: \Omega \to R^{k_{12}}$ such that $F^x = F^{x_1} \vee F^{x_2}$, $(F^{y_{11}},F^{x_1},F^{y_{21}}) \in CIG_{min}$ and $(F^{y_{12}},F^{x_2},F^{y_{22}}) \in CIG_{min}$. With 3.7 one concludes that a basis for x_1 may be chosen such that $x_1 = y_{11} = y_{21}$ a.s., hence $Q_{y_{11}x_1} = I = Q_{y_{21}x_1} = Q_{x_1x_1}$.

It follows from $(F^{y_{21}}, F^{x_2}, F^{y_{22}}) \in CIG_{min}$ and 4.3 that there exists a $Q \in \underline{Q}$ such that $r_1(Q) = (R^{k_{12}}, B_{k_n}, G(0,\Sigma))$. Let $A = (\Lambda^{-1} - \Lambda)^{-1}$, $A^{\frac{1}{2}} \in D^+_{k_{12}}$, $M \in R^{k_{12} \times k_{12}}$

$$M = A^{\frac{1}{2}}(Q - \Lambda)A^{\frac{1}{2}}.$$

Because $Q \in \underline{Q}$ and by 4.4 one has $Q \geq \Lambda > 0$. Thus $M = M^T \geq 0$, and from $Q \in \underline{Q}$ follows that $M - M^2 \geq 0$. From a result in linear algebra one may conclude that there exists $S_1 \in O_{k_{12}}$, $U \in D^{0+}_{k_{12}}$, such that $M = S_1 U S_1^T$. Then

$$0 \geq M^2 - M = S_1(U^2 - U)S_1^T,$$

or $U^2 - U \leq 0$, or if $U = \mathrm{diag}(u_1, \ldots, u_{k_{12}})$ then for $i \in Z_{k_{12}}$, $u_i^2 - u_i \leq 0$, or $u_i \in [0,1]$. Finally, take $S \in O_{k_{12}}/C_{k_{12}}(U)$ corresponding to S_1. Note that one obtains the expression for Q given in 4.2.

3. Injectiveness. Let $(R^{n_1}, B_{n_1}, P_1) = (R^{n_2}, B_{n_2}, P_2) \in WPR$. By the surjectiveness of r one may associate with these (U_1, S_1), $(U_2, S_2) \in WPA$, respectively. By part 2 above and 4.3 one has that the corresponding matrices $Q_1, Q_2 \in \underline{Q}$ satisfy $Q_1 = Q_2$. From the expressions for Q_1, Q_2 it follows that $S_1 U_1 S_1^T = S_2 U_2 S_2^T$. From linear algebra and the fact that the diagonal elements of U_1, U_2 are decreasingly ordered, one deduces that $U_1 = U_2$. Then $S_2^T S_1 U_1 = U_1 S_2^T S_1$. Because $S_1, S_2 \in O_{k_{12}}/C_{k_{12}}(U_1)$ and by definition of $C_{k_{12}}(U_1)$ one concludes that $S_1 = S_2$. □

5. THE STRONG GAUSSIAN PROBABILISTIC REALIZATION PROBLEM

In this section the strong Gaussian probabilistic realization problem is solved.

5.1. <u>DEFINITION</u>. Given a complete probability space (Ω, F, P) and three Gaussian random variables defined on it $y_1 \colon \Omega \to R^{k_1}$, $y_2 \colon \Omega \to R^{k_2}$, $v \colon \Omega \to R^m$ with $(y_1, y_2, v) \in G(0,L)$. Define the *set of strong Gaussian probabilistic realizations*

$$\mathrm{SPR}(R^{k_1+k_2+m}, B_{k_1+k_2+m}, G(0,L))$$

$$= \{(R^n, B_n), F^x \in \underline{F} \mid x\colon \Omega \to R^n.\ F^x = \sigma(\{x\}),$$

$$(F^{y_1}, F^x, F^{y_2}) \in \mathrm{CIG}_{min},\ F^x \subset F^{y_1} \vee F^{y_2} \vee F^v,$$

$$(y_1, y_2, v, x) \in G\}.$$

In the above definition v represents additional information on which the σ-algebra F^x may be based. It is clear that for an arbitrary Gaussian random variable w representing external information, one may construct a Gaussian random variable v such that $F^v \subset F^w$, $(F^v, F^{y_1} \vee F^{y_2}) \in I$, and $F^{y_1} \vee F^{y_2} \vee F^w = F^{y_1} \vee F^{y_2} \vee F^v$.

5.2. THEOREM. *Given a complete probability space* (Ω, F, P) *with three Gaussian random variables defined on it* $y_1\colon \Omega \to R^{k_1}$, $y_2\colon \Omega \to R^{k_2}$, $v\colon \Omega \to R^m$, $(y_1, y_2, v) \in G(0,L)$, *where*

$$L = \begin{pmatrix} K & 0 \\ 0 & I_m \end{pmatrix},$$

with K *as given in* 3.1. *The notation of* 3.1 *is adopted. Let*

$$\mathrm{SPA} = \{(n_1, n_2, n_3) \in N^3_{k_{12}},\ U_3 \in D^{+0}_{n_3},\ S \in O_{k_{12}},\ H \in R^{n_3 \times m} \mid$$

$$n_1 + n_2 + n_3 = k_{12},\ HH^T = I_{n_3},$$

$$U_3 = \mathrm{diag}(u_{31}, \ldots, u_{3n_3}),\ 1 > u_{31} \geq u_{32} \geq \ldots \geq u_{3n_3} > 0,$$

$$\text{if } U = \mathrm{blockdiag}(U_{n_2}, U_3, O_{n_1}) \text{ then } S \in O_{k_{12}}/C_{k_{12}}(U)\}$$

Define the map $r\colon \mathrm{SPA} \to \mathrm{SPR}(R^{k_1+k_2+m}, B_{k_1+k_2+m}, G(0,L))$, $r(n_1, n_2, n_3, U_3, S, H) = (R^n, B_n, F^x)$, *where*

$$n = k_{11} + k_{12},\quad A = (\Lambda^{-1} - \Lambda)^{-1},\quad A^{\frac{1}{2}} \in D^+_{k_{12}},$$

$$U = \mathrm{blockdiag}(I_{n_2}, U_3, O_{n_1}),$$

$$P_1 = A^{-\frac{1}{2}}S(I-U)S^T\Lambda^{\frac{1}{2}}A^{\frac{1}{2}},$$

$$P_2 = A^{-\frac{1}{2}}SUS^T\Lambda^{-\frac{1}{2}}A^{\frac{1}{2}},$$

$$P_3 = A^{-\frac{1}{2}}S(U-U^2)^{\frac{1}{2}}\begin{pmatrix}0\\H\\0\end{pmatrix},$$

$$x\colon \Omega \to R^n, \qquad x = \begin{pmatrix} y_{11} \\ P_1y_{12}+P_2y_{22}+P_3v \end{pmatrix}.$$

Then, with respect to the given basis for (y_1,y_2,v), r *is well defined and a bijection.*

The solution to the strong Gaussian probabilistic realization problem is provided by Theorem 5.2, since it classifies all strong realizations and gives an algorithm to construct all these realizations. The structure of the representation of F^x may be illustrated as follows. Let U = blockdiag $(I_{n_2},U_3,0_{n_1})$ and $x^T = (x_1^T,x_2^T)$ as indicated in 5.2. Then, up to a transformation, the first n_2 components of x_2 consist only of elements of y_{22}, the last n_1 components of x_2 consist only of elements of y_{12}, while the remaining n_3 components consist of elements of y_{21},y_{22} and v jointly.

5.3. LEMMA. *Given* (Ω,F,P) *with three Gaussian random variables defined on it* $y_1\colon \Omega \to R^k$, $y_2\colon \Omega \to R^k$, $v\colon \Omega \to R^m$, $(y_1,y_2,v) \in G(0,L)$,

$$L = \begin{pmatrix} \Sigma & 0 \\ 0 & I_m \end{pmatrix} \in R^{(2k+m)\times(2k+m)},$$

where Σ *is as defined in* 3.1. *Let*

$$\begin{aligned} SPA_1 = \{Q \in R^{k\times k},\ P_3 \in R^{k\times m} \mid\ & A := (\Lambda^{-1}-\Lambda)^{-1}, \\ & Q = Q^T \geq 0,\ Q + Q\Lambda A + A\Lambda Q - QAQ - A = P_3P_3^T\}. \end{aligned}$$

Define the map $r_1\colon SPA_1 \to SPR(R^{2k+m},B_{2k+m},G(0,L))$ *by* $r_1(Q,P_3) = (R^k,B_k,F^x)$, *where*

$$P_1 = (I - Q\Lambda)\Lambda^{\frac{1}{2}}(I-\Lambda^2)^{-1},$$

$$P_2 = (Q-\Lambda)\Lambda^{\frac{1}{2}}(I-\Lambda^2)^{-1},$$

$$x\colon \Omega \to R^k, \quad x = P_1y_1 + P_2y_2 + P_3v, \quad F^x = \sigma(\{x\}).$$

Then, with respect to the given basis for (y_1,y_2,v) *the map* r_1 *is well defined and a bijection.*

PROOF.

1. Some equalities are derived first. Note that Σ is nonsingular and its inverse may be found in the proof of 4.4. Then P_1,P_2 satisfy

$$(P_1 \ \ P_2) = (\Lambda^{\frac{1}{2}}|Q\Lambda^{\frac{1}{2}})\Lambda^{-1},$$

$$(P_1 \ \ P_2)\Sigma\begin{pmatrix}P_1^T\\ P_2^T\end{pmatrix} = QAQ + A - Q\Lambda A - A\Lambda Q.$$

2. To show that r is well defined let $(Q,P_3) \in SPA_1$. Then F^x is well defined. It is then a calculation to show that $Q_{y_1x} = \Lambda^{\frac{1}{2}}$, $Q_{y_2x} = \Lambda^{\frac{1}{2}}Q$, and with 1. above and $(Q,P_3) \in SPA_1$ that

$$0 \le Q_{xx} = (P_1 \ \ P_2)\Sigma\begin{pmatrix}P_1^T\\ P_2^T\end{pmatrix} + P_3P_3^T$$

$$= QAQ + A - Q\Lambda A - A\Lambda Q + P_3P_3^T = Q.$$

Now $Q = Q^T \ge 0$ and

$$Q + Q\Lambda A + A\Lambda Q - QAQ - A = P_3P_3^T \ge 0$$

imply by 4.4 that $Q \ge \Lambda > 0$. Then

$$Q_{y_1y_2} = \Lambda = \Lambda^{\frac{1}{2}}Q^{-1}Q\Lambda^{\frac{1}{2}} = Q_{y_1x}Q_{xx}^{-1}Q_{xy_2},$$

$$\text{rank}(Q_{y_1x}) = \text{rank}(Q_{xx}) = \text{rank}(Q_{y_2x}),$$

and by 3.3, $(F^{y_1}, F^x, F^{y_2}) \in CIG_{min}$. Thus $r_1(Q,P_3) = (R^k, B_k, F^x) \in SPR$.

3. Surjectiveness. Let $(R^k, B_k, F^x) \in SPR$. As in the Proof of 4.3, point 2, one may choose a basis for x such that if $Q = Q_{xx}$, then $Q_{y_1x} = \Lambda^{\frac{1}{2}}$, $Q_{y_2x} = \Lambda^{\frac{1}{2}}Q$. Using 1. above, one obtains

$$E[x|F^{y_1} \vee F^{y_2}] = (\Lambda^{\frac{1}{2}} \quad Q\Lambda^{\frac{1}{2}})\Sigma^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P_1y_1 + P_2y_2.$$

Let $z: \Omega \to R^k$, $z = x - P_1y_1 - P_2y_2$. Then $F^z \subset F^{y_1} \vee F^{y_2} \vee F^v$, z is independent of (y_1,y_2), and $(y_1,y_2,v,z) \in G$, hence there exists a $P_3 \in R^{k\times m}$ such that $z = P_3v$. Furthermore,

$$P_3P_3^T = E[zz^T] = Q - (P_1 \quad P_2)\Sigma\begin{pmatrix} P_1^T \\ P_2^T \end{pmatrix}$$

$$= Q + Q\Lambda A + A\Lambda Q - QAQ - A.$$

Thus $(Q,P_3) \in SPA_1$.

4. Injectiveness. Let $r_1(Q_1,P_{13}) = (R^k, B_k, F^{x_1}) = (R^k, B_k, F^{x_2}) =$
$= r_1(Q_2,P_{23}) \in SPR$. Suppose that for x_1,x_2 a basis has been chosen as in 3. above. Because $F^{x_1} = F^{x_2} \subset F^{y_1} \vee F^{y_2} \vee F^v$, $(y_1,y_2,v,x_1) \in G$ and $(y_1,y_2,v,x_2) \in G$, there exists a nonsingular $S \in R^{k\times k}$ such that $x_2 = Sx_1$. Let

$$x_1 = P_{11}y_1 + P_{12}y_2 + P_{13}v, \quad x_2 = P_{21}y_1 + P_{22}y_2 + P_{23}v.$$

Then

$$P_{21}y_1 + P_{22}y_2 + P_{23}v = SP_{11}y_1 + SP_{12}y_2 + SP_{13}v,$$

$$(SP_{11} - P_{21}) = -(SP_{12} - P_{22})\Lambda,$$

$$(SP_{11} - P_{21})\Lambda = -(SP_{12} - P_{22}), \quad P_{23} = SP_{13}.$$

Using the expressions for $P_{11}, P_{12}, P_{21}, P_{22}$ and performing some calculations, one obtains

$$SP_{11} - P_{21} = [(S-I) - (SQ_1-Q_2)\Lambda]\Lambda^{\frac{1}{2}}(I-\Lambda^2)^{-1}$$

$$SP_{12} - P_{22} = [(SQ_1-Q_2) - (S-I)\Lambda]\Lambda^{\frac{1}{2}}(I-\Lambda^2)^{-1},$$

$$(S-I)(I-\Lambda^2) = 0 \rightarrow S = I,$$

$$(Q_1-Q_2)(I-\Lambda^2) = 0 \rightarrow Q_1 = Q_2, \quad P_{23} = P_{13}. \qquad \square$$

PROOF of 5.2.

1. It will be shown that r is well defined. Given $(n_1,n_2,n_3,U_3,S,H) \in SPA$. A calculation then shows that

$$Q := E[x_2x_2^T] = (P_1 \;\; P_2)\Sigma\begin{pmatrix}P_1^T\\P_2^T\end{pmatrix} + A^{-\frac{1}{2}}S(U-U^2)S^TA^{-\frac{1}{2}}$$

$$= A^{-\frac{1}{2}}SUS^TA^{-\frac{1}{2}} + \Lambda,$$

$$Q_{y_{12}x_2} = \Lambda^{\frac{1}{2}}, \quad Q_{y_{22}x_2} = \Lambda^{\frac{1}{2}}Q,$$

$$Q + Q\Lambda A + A\Lambda Q - QAQ - A \geq 0.$$

By $Q = Q^T \geq 0$, the above inequality, and 4.4, one obtains that $Q \geq \Lambda > 0$. Then (b) of 3.3 follows and thus $(F^{y_1},F^x,F^{y_2}) \in CIG_{min}$. Hence $r(n_1,n_2,n_3,U_3,S,H) = (R^n,B_n,F^x) \in SPR$.

2. Surjectiveness. Let $(R^n,B_n,F^x) \in SPR$. As in point 2 of the Proof of 4.2 it follows that there exists $x_1\colon \Omega \rightarrow R^{k_{11}}$, $x_2\colon \Omega \rightarrow R^{k_{12}}$ such that $(F^{y_{11}},F^{x_1},F^{y_{21}}) \in CIG_{min}$, $x_1 = y_{11}$, $(F^{y_{12}},F^{x_2},F^{y_{22}}) \in CIG_{min}$, and $F^x = F^{x_1} \vee F^{x_2}$. Then $(F^{y_{12}},F^{x_2},F^{y_{22}}) \in CIG_{min}$ and 5.3 imply that there exists $(Q,P_3) \in SPA_1$ such that $r_1(Q,P_3) = (R^{k_{12}},B_{k_{12}},F^{x_2})$. Then

$$Q + Q\Lambda A + A\Lambda Q - QAQ - A = P_3P_3^T,$$

and with $M \in R^{k_{12}\times k_{12}}$, $M := A^{\frac{1}{2}}(Q-\Lambda)A^{\frac{1}{2}}$,

$$P_3P_3^T = A^{-\frac{1}{2}}(M-M^2)A^{-\frac{1}{2}} \geq 0, \quad M = M^T \geq 0.$$

Because $M = M^T \geq 0$, there exists $U \in D^{0+}_{k_{12}}$, $S_1 \in O_{k_{12}}$, such that $M = S_1 U S_1^T$. Let $S \in O_{k_{12}}/C_{k_{12}}(U)$ be the element corresponding to S_1. Let $(n_1,n_2,n_3) \in N^3_{k_{12}}$ be respectively the number of diagonal elements of U that are in $\{0\},\{1\}$, $(0,1)$. Note that because the diagonal elements of U are by convention decreasingly ordered, one has the decomposition $U = \text{blockdiagonal}(I_{n_2}, U_3, 0_{n_1})$, where $U_3 = \text{diag}(u_{31},\ldots,u_{3n_3})$ with $1 > u_{31} \geq u_{32} \geq \ldots \geq u_{3n_3} > 0$. Let $V_1 \in R^{n_2 \times m}$, $V_2 \in R^{n_3 \times m}$, $V_3 \in R^{n_1 \times m}$,

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = S^T A^{\frac{1}{2}} P_3.$$

Then $P_3 P_3^T = A^{-\frac{1}{2}}(M-M^2)A^{-\frac{1}{2}} = A^{-\frac{1}{2}} S(U-U^2) S^T A^{-\frac{1}{2}}$, implies that $V_1 = 0$, $V_3 = 0$, $V_2 V_2^T = U_3 - U_3^2$. Take $(U_3-U_3^2)^{-\frac{1}{2}} \in D^+_{n_3}$, $H = (U_3-U_3^2)^{-\frac{1}{2}} V_2 \in R^{n_3 \times m}$. Then $HH^T = I_{n_3}$.

3. Injectiveness. Let $r(n_{11},n_{12},n_{13},U_{13},S_1,H_1) = r(n_{21},n_{22},n_{23},U_{23},S_2,H_2) \in SPR$. As indicated in 2. above there exists $(Q_1,P_{13}) = (Q_2,P_{23})$ corresponding to these elements. From the expression

$$Q_1 = A^{-\frac{1}{2}} S_1 U_1 S_1^T A^{-\frac{1}{2}} + \Lambda$$

and a similar expression for Q_2, it follows that $S_1 U_1 S_1^T = S_2 U_2 S_2^T$. Because of a result in linear algebra and the convention of $D^{0+}_{k_{12}}$, one obtains that $U_{13} = U_{23}$ and $n_{11} = n_{21}$, $n_{12} = n_{22}$, $n_{13} = n_{23}$. Furthermore, $S_2^T S_1 U_{13} = U_{13} S_2^T S_1$ and $S_1, S_2 \in O_{k_{12}}/C_{k_{12}}(U_{13})$ imply that $S_1 = S_2$. Finally,

$$A^{-\frac{1}{2}} S_1 (U_{13}-U_{13}^2)^{\frac{1}{2}} \begin{pmatrix} 0 \\ H_1 \\ 0 \end{pmatrix} = P_{13} = P_{23} = A^{-\frac{1}{2}} S_2 (U_{13}-U_{13}^2)^{\frac{1}{2}} \begin{pmatrix} 0 \\ H_2 \\ 0 \end{pmatrix}$$

imply that $H_1 = H_2$. □

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